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Multivariate Exponential Distributions with Constant Failure Rates

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In this paper a multivariate failure rate representation based on Cox's conditional failure rate is introduced, characterizations of the Freund–Block and the Marshall–Olkin multivariate exponential distributions are obtained, and generalizations of the Block–Basu and the Friday–Patil bivariate exponential distributions are proposed. © 1997 Academic Press

1. INTRODUCTION

Unlike the multivariate normal distribution, there are many bivariate exponential distributions and most of them have been extended to the n-dimensional case. A review of the literature is given by Basu [3]. The failure rate of a univariate random variable plays an important role in the probability theory of reliability and survival analysis and it is well known that a constant failure rate is a fundamental characterization of the univariate exponential distribution. Although many characterization results for the multivariate exponential distribution have been obtained during the past two decades (see, for example, Azlarov and Volodin [1]), characterizations based on properties analogous to a constant failure rate have not been fully explored. Extension of the univariate failure rate concept to

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n dimensions is not straightforward. Attempts at such an extension have been made by, for example, Basu [2], Puri and Rubin [14], Johnson and Kotz [10], and Marshall [11]. However, most multivariate exponential distributions do not have constant failure rate(s) according to concepts defined before.

Cox [6] introduces a concept of conditional failure rate and a failure rate formulation for the absolutely continuous bivariate variable which view the bivariate lifetime as a point process. Cox's formulation is very useful and has been extended to multivariate failure processes, e.g., Shaked and Shanthikumar [15]. However, little work has been done on characterizing multivariate exponential distributions by using this formulation. Sun and Basu [16] apply Cox's conditional failure rate to the case where a singular part exists in a bivariate distribution or, in terms of system reliability, simultaneous failures may occur in a two-component system, and obtain characterizations of the bivariate exponential distributions of the BEE [8] family. In this paper, we extend the results to the n-dimensional case. Properties and characterizations of the Freund–Block [4] and the Marshall–Olkin [12] n-dimensional exponential distributions are discussed in Section 2. Generalizations of the Block–Basu [5] and the Friday–Patil [8] bivariate exponential distributions are proposed in Section 3.

Failure of an n-component parallel system can be considered to consist of n possible stages: one (or more) of the n components fails first, then one (or more) of the remaining components fails, and so on. In a situation where one of the components has failed, it is useful to take advantage of that knowledge when considering the still surviving component's residual life. From this point of view, it seems reasonable to represent the failure rate of an n-component system by using the first stage failure rate, that is, the failure rate of the minimum lifetime of the n components, the second stage failure rate, that is, the conditional failure rate of the minimum residual lifetime of the still surviving component(s) given that one (or more) component fails first, and so on. In fact, failures of n components are equivalent to a pair of ranks and order of those failure times. Stage-by-stage failure times are the concepts useful to describe the ranks of the failure times. Orders can be captured using ratios of the failure rate of the failing component to the sum of those failure rates of the risk sets. To this end we define the total failure rate of a multivariate random variable as follows.

Let $(X_1, X_2, ..., X_n)$ be a non negative n-dimensional random variable having joint survival function $\bar{F}(x_1, x_2, ..., x_n) = P\{X_1 > x_1,\ X_2 > x_2, ..., X_n > x_n\}$ with $\bar{F}(0, 0, ..., 0) = 1$. Let ς be the class of nonempty subsets of $\{1, 2, ..., n\}$, $R_K = \{i_k, i_{k+1}, ..., i_n\} \subset \{1, 2, ..., n\}$ for $k = 1, 2, ..., n$. Where R_K is called risk set at the kth stage, indices of components still surviving (therefore, facing risks of failure) just before the kth failure and $i_1, ..., i_n$, are observation indices defined such that $X_{i_1} < \cdots < X_{i_n}$. Let $D_k = \{1, 2, ..., n\} - R_K$ and D_k be called the death set representing the indices of components which have failed by prior to the kth failure. Let $X_{R_K} = (X_{i_k}, ..., X_{i_n})$. Let $X_{R_K} > X_{D_k}$ denote $\min\{X_i: i \in R_K\} > \max\{X_i: i \in D_k\}$, for $D_k \neq \varnothing$, and $\min\{X_i: i \in R_K\} > 0$ for $D_k = \varnothing$. Let $X_{R_K} > t$ denote $\min\{X_i: i \in R_K\} > t$, where $t \in [0, \infty)$.

DEFINITION 1. If $\bar{F}(x_1, x_2, ..., x_n)$ is absolutely continuous on $x_i \neq x_j$, $i \neq j$, the vector

$$(r_{R_K, D_k}(t \mid x_{D_k}), \text{ for } x_{R_K} > x_{D_k},\ R_K = \{i_k, i_{k+1}, ..., i_n\},\ k = 1, ..., n.)$$

is called the total failure rate of $(X_1, X_2, ..., X_n)$ or of $\bar{F}$, where

$$r_{R_1, D_1}(t \mid x_{D_1}) = -d \log P\{X_1 > t, X_2 > t, ..., X_n > t\}/dt,$$

and for $(k = 2, ..., n)$,

$$r_{R_k, D_k}(t \mid x_{D_k}) = -d \log P\{X_{R_K} > t \mid X_{D_k} = x_{D_k}\}/dt, \qquad \text{for} \quad D_k \neq \varnothing. \tag{1.1}$$

For $k = 1$, D_1 is a null set, and we denote $r_{R_1, D_1}(t \mid x_{D_1})$ by $r(t)$. For $k = 2, ..., n$, D_k is not empty, and $r_{R_K, D_k}(t \mid x_{D_k})$ is the conditional failure rate of $\min\{X_i: i \in R_K\}$ given $X_{R_K} > X_{D_k}$ and $X_{D_k} = x_{D_k}$.

2. CHARACTERIZATIONS OF TWO MULTIVARIATE EXPONENTIAL DISTRIBUTIONS

In this section, we first discuss properties of two multivariate exponential distributions and then characterize these distributions in terms of total failure rate.

The Freund bivariate exponential distribution [7] has been extended by Block [4] to the n-dimensional case. We call it the Freund–Block distribution. The joint density is

$$f(x_1, x_2, ..., x_n) = \left\{\prod_{k=1}^{n} \alpha^{(k)}_{i_k, \{i_1, ..., i_{k-1}\}}\right\}\left\{\exp\left[-\sum_{k=1}^{n}\left(\sum_{j=k}^{n} \alpha^{(k)}_{i_j, \{i_1, ..., i_{k-1}\}}(x_{i_k} - x_{i_{k-1}})\right)\right]\right\}, \tag{2.1}$$

for $0 = x_{i_0} < x_{i_1} < x_{i_2} \cdots < x_{i_n}$, where $\alpha^{(k)}_{i_j, \{i_1, ..., i_{k-1}\}} > 0$.

According to Definition 1, this distribution has a constant total failure rate,

$$\left(\sum_{j=1}^{n} \alpha_{i_j}^{(1)}; \sum_{j=2}^{n} \alpha_{i_j,\{i_1\}}^{(2)}; \ldots; \alpha_{i_n,\{i_1,\ldots,i_{n-1}\}}^{(n)}\right), \tag{2.2}$$

where $\alpha_j^{(0)} = \alpha_{j,\{i_0\}}^{(0)}$.

Marshall and Olkin [12] propose a multivariate exponential distribution (MVE) which has the joint survival function

$$\begin{aligned}\bar{F}&(x_1, x_2, \ldots, x_n)\\ &= \exp\Bigg[-\sum_{i=1}^{n} \lambda_i x_i - \sum_{i<j} \lambda_{ij} \max(x_i, x_j)\\ &\quad - \sum_{i<k<j} \lambda_{ijk} \max(x_i, x_j, x_k) - \cdots - \lambda_{12\ldots n} \max(x_1, x_2, \ldots, x_n)\Bigg]\end{aligned} \tag{2.3}$$

for $x_1, x_2, \ldots, x_n \geqslant 0$.

Using the notation of Block [4], let $\varsigma(R_k) = \{J \in \varsigma : R_k \mid J \neq \varnothing\}$, which is the class of sets of component indices having at least one index common with R_k, the set of surviving components at the kth stage. Let $\lambda_J = \lambda_{j_1 j_2 \cdots j_k}$, where $J = \{j_1, j_2, \ldots, j_k\}$ and either quantity does not depend on the order of the j_i. The survival function (2.3) can be written

$$\bar{F}(x_1, x_2, \ldots, x_n) = \exp\left[-\sum_{k=1}^{n}\left(\sum_{J \in \varsigma(R_k)} \lambda_J\right)(x_{i_k} - x_{i_{k-1}})\right], \tag{2.4}$$

for $0 = x_{i_0} \leqslant x_{i_1} \leqslant x_{i_2} \leqslant \cdots \leqslant x_{i_n}$, where $\lambda_J > 0$.

To better understand (2.3) and (2.4), the underlying fatal shock model, which motivates the MVE, can be considered: there are $2^n - 1$ underlying fatal shocks arriving independently according to Poisson shock processes with intensity rates λ_J's where J's are nonempty subsets of $\{1, \ldots, n\}$ with $\#(J) = 2^n - 1$. For instance, when $J = \{j_1, \ldots, j_k\}$, the corresponding shock arrives according to a Poisson process with intensity rate $\lambda_{\{j_1, \ldots, j_k\}}$ and inflicts fatal damages simultaneously to those k componenets indexed by $j_1, j_2, \ldots, {}_k$. Therefore, the failure rate of the jth component is the sum of all failure rates corresponding to shocks which damage the jth component, that is, $\sum_{J \in \varsigma(j)} \lambda_J$. As a result, the total failure rate at the kth stage is the sum of all shock arrival rates of shocks involving any of the ten surviving components, that is, $\sum_{J \in \varsigma(R_K)} \lambda_J$.

According to Definition 1, this distribution has a constant total failure rate of

$$\left(\sum_{J \in \varsigma(R_1)} \lambda_J, ..., \sum_{J \in \varsigma(R_1)} \lambda_J\right). \tag{2.5}$$

The above two distributions can be derived from several physical models, respectively. Derivations based on constant total failure rate, however, are often intuitively appealing and may shed light on the applicability of these distributions. The following theorem is easily verified.

THEOREM 2.1 *Suppose that* $(X_1, X_2, ..., X_n)$ *is a nonnegative n-dimensional random variable.*

(a) $(X_1, X_2, ..., X_n)$ *is the* MVE *if and only if it has a total failure rate given by* (2.2), *and*

$$P\{\min(X_i: i \in R_K) = X_{i_j} \mid X_{R_K} > X_{D_k}, X_{D_k} = x_{D_k}, \min(X_i: i \in R_K) = t\}$$

$$= \alpha_{i_j, D_k}^{(k-1)} \Big/ \sum_{i \in R_k} \alpha_{i, D_k}^{(k-1)}, \tag{2.6}$$

then $(X_1, X_2, ..., X_n)$ *has the Freund–Block distribution*;

(b) $(X_1, X_2, ..., X_n)$ *is the* MVE *if and only if it has a total failure rate given by* (2.5) *and*

$$P\{\min(X_m: m \in R_K) = X_{m_1} = \cdots = X_{m_j} \mid X_{R_K} > X_{D_k},$$
$$X_{D_k} = x_{D_k}, \min(X_i: i \in R_K) = t\}$$
$$= \sum_{J \in \varsigma'(D_k \cup \{m_1, ..., m_j\})} \lambda_J \Big/ \sum_{J \in \varsigma(R_k)} \lambda_J, \tag{2.7}$$

where $\varsigma'(D_k \cup \{m_1, ..., m_j\}) = \{m_1, ..., m_j \cup J: J \in \varsigma(D_k)\}$.

The loss of memory property (LMP) of Marshall and Olkin [12] is given by the equation

$$\bar{F}(x_1 + t, x_2 + t, ..., x_n + t) = \bar{F}(x_1, x_2, ..., x_n)\, \bar{F}(t, t, ..., t). \tag{2.8}$$

for all $t > 0$ and $x_i > 0$, $i = 1, ..., n$.

As in the bivariate case, under appropriate conditions the constant total failure rate property implies the LMP. To prove the following theorem we extend the notation of Proschan and Sullo [13] to a general setup.

Let $\mathfrak{R}$ denote the class of all possible ordered set of $X_1, X_2, ..., X_n$. For example, if

$$n=3, \qquad \mathfrak{R}=\{X_1<X_2<X_3, ..., X_3<X_2<X_1, X_1=X_2<X_3, ..., \\ X_3=X_2<X_1, X_1=X_2=X_3\}. \tag{2.9}$$

For $0=x_{i_0} \leqslant x_{i_1} \leqslant x_{i_2} \leqslant \cdots \leqslant x_{i_n}$, let $t_j=x_{i_j}-x_{i_{j-1}}$, $j=1, ..., n$.

Let $\chi[\cdot]$ denote the set characteristic function. Then the distribution function of $\chi[R]$, $R\in\mathfrak{R}$, is multinomial $(1, P\{R\}: R\in\mathfrak{R})$. Let μ be the Lebesgue measure on $[0, \infty)$, $\mu_i=\mu\chi(t_i\neq 0)+\delta_i\chi(t_i=0)$, where δ_i denotes the degenerate measure at $t_i=0$. The joint density of $(T_1, T_2, ..., T_n, \chi[R])$ can be written with respect to the product measure $\mu_1\times\mu_2\times\cdots\times\mu_n\times\eta$, where η denotes the counting measure on the sample space of a multinomial random variable with parameters $(1, P\{R\}: R\in\mathfrak{R})$.

Theorem 2.2. *If $(X_1, X_2, ..., X_n)$ has a constant total failure rate and for $R_K\in\varsigma$ the quantities*

$$P\{\min(X_m: m\in R_K)=X_{m_1}=\cdots=X_{m_j} \mid X_{R_K}>X_{D_k}, \\ X_{D_k}=x_{D_k}, \min(X_i: i\in R_K)=t\},$$

are constant, then the LMP *holds.*

Proof. Let the total failure rate be $(r_{R_K, D_k}, R_K\in\varsigma)$. From the assumption we know

$$P\{\min(X_m: m\in R_K)=X_{m_1}=\cdots=X_{m_j} \mid X_{R_K}>X_{D_k}, \\ X_{D_k}=x_{D_k}, \min(X_i: i\in R_K)=t\} \\ =P\{\min(X_m: m\in R_K)=X_{m_1}=\cdots=X_{m_j} \mid X_{R_K}>X_{D_k}\}, \qquad R_K\in\varsigma,$$

and $P\{R\}$ is a product of some of these conditional probabilities.

Using the transformation (2.9) from the product measure $\mu_1\times\mu_2\times\cdots\times\mu_n\times\eta$, the density, with respect to the induced measure, can be written

$$f(x_1, x_2, ..., x_n)=\prod_{R\in\mathfrak{R}} [P\{R\}]^{\chi[R]} \prod_{i=1}^{n} r_{\{n, ..., i\}, \{i-1, ..., 1\}}^{\delta(x_i-x_{i-1})} \\ \times\exp[-r_{\{n, ..., i\}, \{i-1, ..., 1\}}(x_i-x_{i-1})], \tag{2.10}$$

where $\delta(x_i-x_{i-1})=1$, if $x_i-x_{i-1}>0$, and $=0$, if $x_i-x_{i-1}=0$, for $0=x_0\leqslant x_1\leqslant x_2\leqslant\cdots\leqslant x_n$.

The quantity (2.10) can be understood using the decomposition of stage-by-stage conditional failure rates: the first product term captures order information, and the second captures rank information.

The density for $0 = x_{i_0} \leqslant x_{i_1} \leqslant x_{i_2} \leqslant \cdots \leqslant x_{i_n}$ can be obtained for substituting subsets of $\{1, ..., n\}$ in (2.10) by corresponding subsets of $\{i_1, i_2, ..., i_n\}$.

From the functional form of the density (2.10), we have

$$\begin{aligned} f(x_1+t, x_2+t, ..., x_n+t) & \\ &= \exp(-r \cdot t)\, f(x_1, x_2, ..., x_n) \\ &= P\{X_1 > t, X_2 > t, ..., X_n > t\}\, f(x_1, x_2, ..., x_n), \qquad t > 0. \end{aligned}$$

Upon integration the LMP is obtained. ∎

We are now ready to discuss characterizations of the above two multivariate distributions. Unlike the bivariate case, and absolutely continuous distribution with both constant total failure rate and constant quantities

$$P\{\min(X_i\colon i \in R_K) = X_{i_j} \mid X_{R_K} > X_{D_k}, X_{D_k} = x_{D_K}, \min(X_i\colon i \in R_K) = t\}$$

may not be the Freund–Block extension. However, we can easily obtain the following theorem.

THEOREM 2.3. *$(X_1, X_2, ..., X_n)$ has the Freund–Block exponential distribution if and only if*

(a) *it has constant $r(t)$, constant $P\{\min(X_i\colon i \neq j) > X_j \mid \min(X_1, ..., X_n) = t\}$ where the summation of these over $j = 1, ..., n$ is 1.*

(b) *given $\min(X_i\colon i \neq j) > X_j = x_j$, the conditional distribution of $(X_i\colon i \neq j)$ is the $(n-1)$-dimensional Freund–Block distribution, for all j.*

To obtain a characterization of the MVE distribution, we consider the following lemma.

LEMMA 2.1. *If $(X_1, X_2, ..., X_n)$ has a constant total failure rate and all $(n-1)$-dimensional marginals are* MVE, *then for all $R_K \in \varsigma$, the quantities*

$$\begin{aligned} P\{\min(X_1\colon 1 \in R_K) = X_{1_1} = \cdots = X_{1_j} \mid X_{R_K} > X_{D_K}, & \\ X_{D_K} = x_{D_K}, \min(X_i\colon i \in R_K) = t\} & \\ = g_{\min(X_1\colon 1 \in R_k) = X_{1_1} I X_{R_k} > X_{D_k}}(x_{D_k}, t) & \end{aligned}$$

are free of both x_{D_k} and t.

Proof. Since the detailed proof is very tedious, to save space we shall just sketch the essential ideas. Let the total failure rate be $(r_{R_K, D_k}, R_K \in \varsigma)$. We can write the density function in terms of g and r_{R_K, D_K}. Integrating the density with respect to x_j, $j = 1, ..., n$, we obtain the $(n-1)$-dimensional marginals. Thus from the assumption that the marginals are MVE, we obtain n functional equations. Differentiating these functional equations respectively, we have a group of differential equations of $g_{\min(X_1 : 1 \in R_k) = X_{1_1} = \cdots = IX_{R_k} > X_{D_k}}(X_{D_k}, t)$. The solutions of these differential equations will be of the form $\sum a_i \exp(-b_i x_{D_k} - c_i t)$, where some b_i and c_i are negative. Since $0 \leqslant g \leqslant 1$ for any (x_{D_k}, t) the result follows. ∎

THEOREM 2.4. *$(X_1, X_2, ..., X_n)$ is* MVE *if and only if $(X_1, X_2, ..., X_n)$ and all its k-dimensional marginals have constant total failures rates, $k = 1, ..., n-1$.*

Proof. The "only if" part follows from the property of MVE. We now prove the "if" part by induction. We know that it is true for $n = 2$. Suppose that it is true for $n-1$. Thus all $(n-1)$-dimensional marginals are MVE. From Lemma 2.1 and Theorem 2.2, it follows that the LMP holds from $(X_1, X_2, ..., X_n)$. Hence $(X_1, X_2, ..., X_n)$ has the n-dimensonal MVE.

Another way of stating the above theorem is given by the following corollary.

COROLLARY 2.1. *The random variable $(X_1, X_2, ..., X_n)$ is* MVE *if and only if $(X_1, X_2, ..., X_n)$ has a constant total failure rate and all its $(n-1)$-dimensional marginals are* MVE.

3. GENERALIZATIONS OF TWO BIVARIATE EXPONENTIAL DISTRIBUTIONS

Block and Basu in [5] propose an exponential bivariate distribution (ACBVE), which is the absolutely continuous part of the BVE as well as a special case of the Freund distribution. Block in [4] extends the ACBVE to the multivariate case (ACMVE), and Hanagal [9] obtains some inference results in ACMVE. The joint density for the ACMVE is

$$f(x_1, x_2, ..., x_n) = (a)^{-1} \prod_{k=1}^{n} (\theta_{R_k} - \theta_{R_{k+1}}) \exp\left[-\sum_{k=1}^{n} \theta_{R_k}(x_{i_k} - x_{i_{k-1}}) \right], \tag{3.1}$$

for $0=x_{i_0}<x_{i_1}<x_{i_2}<\cdots<x_{i_n}$, where $\theta_{R_k}=\sum_{J\in\varsigma(R_1)}\lambda_J$, for $k=1, 2, ..., n$, $\theta_{R_{n+1}}=0$, $R_{n+1}=\varnothing$, $\theta_{R_k}-\theta_{R_{k+1}}=\sum_{J\in\varsigma(R_k)-\varsigma(R_{k+1})}\lambda_J$, $a=\sum'\prod_{k=1}^{n-1}[(\theta_{R_k}-\theta_{R_{k+1}})/\theta_{R_k}]$, and $\sum'$ indicates the summation over all permutation of 1, 2, ..., n.

This distribution has a constant total failure rate

$$(\theta_{R_k}, R_K\in\varsigma). \tag{3.2}$$

The derivation of the ACMVE in Block [4] is given by differentiating the MVE survival function. Another extension of the ACBVE by using the approach in Theorem 2.1 also seems appealing. This makes some properties of the derived distribution more transparent.

Suppose that $(X_1, X_2, ..., X_n)$ has a total failure rate $(\theta_{R_k}, R_K\in\varsigma)$, and

$$P\{\min(X_i: i\in R_K)=X_{i_j}\mid X_{R_K}>X_{D_K}, X_{D_K}=x_{D_K}, \min(X_i: i\in R_K)=t\}$$

$$=(\theta_{R_k}-\theta_{R_{k+1}})/\textstyle\sum^{(k+1)}(\theta_{R_k}-\theta_{R_{k+1}}), \tag{3.3}$$

where $\sum^{(k+1)}$ indicates the summation over all possible $i_j\in R_K$. Then the density function of $(X_1, X_2, ..., X_n)$ will be

$$f(x_1, x_2, ..., x_n)=\left(\prod_{k=1}^{n}\theta_{R_k}\right)\left[\prod_{k=1}^{n-1}(\theta_{R_k}-\theta_{R_{k+1}})/\textstyle\sum^{(k+1)}(\theta_{R_k}-\theta_{R_{k+1}}\right]$$

$$\times\exp\left[-\sum_{k=1}^{n}\theta_{R_k}(x_{i_k}-x_{i_{k-1}})\right] \tag{3.4}$$

for $0=x_{i_0}<x_{i_1}<x_{i_2}<\cdots<x_{i_n}$.

The idea here is to take out the singular part of the MVE stage by stage. This distribution, called ACMVE*, appears to have a good pattern or a natural structure from the above point of view.

Certain constraints on the parameters in the Freund–Block distribution,

$$\theta_{R_k}=\sum_{i=k}^{n}\alpha_i, \qquad k=1, ..., n, \qquad R_K\in\varsigma, \tag{3.5}$$

and

$$\alpha_{i_j, D_k}^{(k-1)}\Big/\sum_{i\in R_k}\alpha_{i, D_k}^{(k-1)}=(\theta_{R_k}-\theta_{R_{k+1}})/\textstyle\sum^{(k+1)}(\theta_{R_k}-\theta_{R_{k+1}}), \tag{3.6}$$

also give the ACMVE* as a special case.

Here we shall point out a special case of the Freund–Block distribution. The distribution given by (2.1) is a complete generalization of the Freund distribution, which can be derived from a fatal shock model. Consider an n-component system. Independent nonhomogeneous Poisson processes govern the the occurrence of fatal shocks. The Freund–Block distribution is derived in Block [4] by assuming that there are $(n-j)\binom{n}{j}$ classes of these processes $\{Z^{(j)}_{k \mid i_1, ..., i_j}(t) \colon k,\ i_1, ..., i_j$ are $j+1$ distinct elements of $1, ..., n\}$ for $j=0, 1, ..., n-1$. However, it seems likely that in some cases the processes are independent of not only the order of $i_1, ..., i_j$ but also the elements of $i_1, ..., i_j$. That is, there are just n classes of these processes for $j=0, 1, ..., n-1$. Then this distribution has n^2 parameters and is given by

$$f(x_1, x_2, ..., x_n)$$
$$=\left\{\prod_{k=1}^{n} \alpha_{i_k}^{(k-1)}\right\}\left\{\exp\left[-\sum_{k=1}^{n}\left(\sum_{j=k}^{n}\alpha_{i_j}^{(k-1)}\right)(x_{i_k}-x_{i_{k-1}})\right]\right\} \tag{3.7}$$

for $0=x_{i_0}<x_{i_1}<x_{i_2}< \cdots <x_{i_n}$.

Friday and Patil [8] propose a bivariate exponential distribution (BEE) which is derivable from a threshold model, a gestation model, and a warmup model. Sun and Basu [16] have shown that among the bivariate exponential distributions with constant total failure rates and constant $P\{X>Y \mid \min(X, Y)=t\}$ the BEE is the largest family. In this direction, it seems appropriate that an extension of the BEE, called the MEE, is defined by an n-dimensional distribution with a constant total failure rate and constant

$$P\{\min(X_m \colon m \in R_K)=X_{m_1}= \cdots =X_{m_j} \mid X_{R_K}>X_{D_K},$$
$$X_{D_K}=x_{D_K}, \min(X_i \colon i \in R_K)=t\}.$$

This density function has been given by (2.10). It is also clear that, as in the bivariate case, with certain constraints on these constants or parameters, the MEE distribution reduces to the MVE, the ACMVE, the ACMVE*, and the Freund–Block distribution, repectively.

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